International Math Battle Tournament

Bovec, 2026

Day 1

1. Prove that there is a positive integer n with the following property: for every integer $1 \leq k \leq n-1$ there exists a prime p (which could depend on k) such that the binomial coefficient $\binom{n}{k}$ is divisible by p^{1000}.

2. Let n be a positive integer. In how many ways can we mark cells on an $n \times n$ board such that no two rows and no two columns have the same number of marked cells?

3. Determine all functions $f\colon \mathbb{R}^+ \to \mathbb{R}$, such that

   - For all $x, y \in \mathbb{R}^+$:
   $$f(x) + f\left(\frac{y}{x}\right) \leq \frac{x^3}{y^2} + \frac{y}{x^3}$$
   - for every $x \in \mathbb{R}^+$, there exists a $y \in \mathbb{R}^+$, such that
   $$f(x) + f\left(\frac{y}{x}\right) = \frac{x^3}{y^2} + \frac{y}{x^3}$$

4. Let ABC be a triangle such that $AB = AC$ and points D and E on line BC such that D, B, C, and E lie on this order. The altitudes from D to AB and from E to AC intersect in F. Let AM and AN be diameters of the circumcircles of $\triangle ACD$ and $\triangle ABE$, respectively.

   Show that the line AF bisects the segment MN.

5. A polynomial $f \in \mathbb{R}[x]$ is called *whole*, if for each polynomial $h(x) \in \mathbb{R}[x]$ there exists a positive integer k and polynomials $g_1(x), g_2(x), \ldots, g_k(x) \in \mathbb{R}[x]$ such that
   $$h(x) = f(g_1(x)) + \ldots + f(g_k(x)).$$
   Determine all whole polynomials.

6. Let $ABCD$ be a cyclic quadrilateral with diameter AC and let X be a point in the extension of ray CA beyond point A. Let Y and Z be the second intersections of lines BA and DA with the circumcircles of $\triangle XAD$ and $\triangle XAB$, respectively.

   Prove that $\angle CYA = \angle AZC$.

7. Find all prime numbers p for which the number
   $$3^p + 4^p + 5^p + 9^p - 98$$
   has at most 6 positive divisors.

8. A subset of the positive integers is called *Han*, if it contains two disjoint, nonempty subsets with equal sum. Let $n \geq 3$ be a positive integer. Show that if A is a set of 2^n positive integers, which don't exceed 2^{2^n-n-1}, then A is Han.

Battle begins at 14:15.

Each problem is worth 12 points.

The use of electronic devices and literature is strictly prohibited.