International Math Battle Tournament

Bovec, 2026

Day 2

1. Let $N = 99\ldots9909$ be the 20-digit integer formed by writing the digit 9 eighteen times, then the digit zero and finally the digit 9 once. Determine the smallest positive integer k such that N can be written as a sum of k palindromes, each of which has at least two digits.

   Remark: A positive integer $n = \overline{a_1a_2\ldots a_k}$ *is a palindrome if* $a_i = a_{k+1-i}$ *for all* $i \leq k$.

2. Let ABC be a triangle with $AB \neq AC$ and D a point on BC such that AD bisects $\angle BAC$. Let E be on side BC such that $BD = CE$. Let F be a point on the arc BC (that does not contain point A) such that $\angle BAF = \angle CAE$. The line parallel to DF passing through E intersects line AF at point X.

   Prove that $AC \cdot BX = AB \cdot CX$.

3. There are $n \geq 3$ chameleons sitting on a circle, some being green, some orange and some purple. Every second, Koza chooses a pair of neighboring chameleons of different colors and changes them into the third color.

   Find all integers n such that Koza can eventually turn all chameleons green regardless of their original colors.

4. Let p be a prime number greater than 5. Let

   $$1 = a_1 < a_2 < \cdots < a_{\frac{p-1}{2}}$$

   be all the distinct non-zero quadratic residues modulo p in increasing order. Consider the sequence of partial products

   $$a_1,\ \ a_1a_2,\ \ a_1a_2a_3,\ \ \ldots,\ \ a_1a_2\cdots a_{\frac{p-1}{2}}.$$

   Prove that for every prime $p > 5$, at least two of these products are equal modulo p.

5. Positive real numbers $x_1, x_2, x_3, \ldots, x_{2025}$ are written on a blackboard. A move consists of selecting two numbers a and b on the board, erasing them, and writing the number

   $$\frac{a^2 + 6ab + b^2}{a + b}$$

   on the blackboard. After 2024 moves, only one number c will remain on the board. Prove that

   $$c < 2025(x_1 + x_2 + \ldots + x_{2025}).$$

6. Let $\triangle ABC$ be an acute triangle and D, E, F points on sides BC, CA, AB, respectively, and distinct from the vertices of triangle. Let the circumcircles of $\triangle BDF$ and $\triangle CDE$ intersect again at point G.

   Prove that, as D moves along segment BC, the line DG passes through a fixed point.

7. Find all functions $f\colon \mathbb{Z}^+ \to \mathbb{Z}^+$, such that for all positive integers m and n,

$$f\Big(m \cdot d(m) + n^2\Big) = f(m) \cdot d(m) + n \cdot f(n),$$

   where $d(x)$ denotes the number of positive divisors of x.

8. There is an urn containing 300 green balls labeled with the number 2, 200 jade balls labeled with the number 3, and 100 violet balls labeled with the number 6. Gabe, John, and Vini play a game with 600 rounds (in each round, only one player participates). The rules are as follows:

   - Initially, all the balls are drawn one by one from the urn in a random order. The drawn ball is chosen uniformly and independently of previously drawn balls.
   - Gabe will play in the i-th round if and only if the i-th ball drawn is green.
   - John will play in the i-th round if and only if the i-th ball drawn is jade.
   - Vini will play in the i-th round if and only if the i-th ball drawn is violet.
   - In each round, the designated player must choose a positive real number strictly less than the number written on the corresponding ball.
   - The winner is the player who obtains the greatest sum of the numbers they chose during their rounds.

   What is the probability that the random order of the balls guarantees a winning strategy for each of them?

Battle begins at 14:15.

Each problem is worth 12 points.

The use of electronic devices and literature is strictly prohibited.