International Math Battle Tournament

Bovec, 2026

Day 3

1. Let x, y and z be positive real numbers such that $xy + yz + zx = x + y + z$. Prove that

$$\frac{1}{x^2 + y + 1} + \frac{1}{y^2 + z + 1} + \frac{1}{z^2 + x + 1} \leq 1$$

and determine when equality holds.

2. Let ABC be a triangle and Ω be a circle touching AB and AC. Let ω_1 be the circle inside the angle $\angle CBA$ that touches Ω, AB and BC at three distinct points. Similarly, let ω_2 be the circle inside the angle $\angle ACB$ that touches Ω, AC and BC at three distinct points. Let the circle ω_1 touch Ω at point P and the side BC at Q. Similarly, let the circle ω_2 touch Ω at point R and side BC at point S.

   Prove that the points P, Q, R and S lie on a circle.

3. Let I and Ω be the incenter and circumcircle of an acute non-isosceles triangle ABC. Let BI and CI intersect the altitude of ABC through A at U and V, respectively. The circle with diameter AI intersects Ω again at T, and the circumcircle of triangle TUV intersects the segment BC and Ω at P and Q, respectively. Let R be the other intersection of PQ and Ω.

   Prove that $AR \parallel BC$.

4. An epidemic is raging in a pond with n ducks, where n is a positive integer. Some ducks are friends, such that is A is a friend of B, then B is also a friend of A. Each duck can be either *healthy*, *sick*, or *recovering*. The first day, some ducks get sick, while the rest are healthy. On each day, healthy friends of ducks who were sick on the previous day get sick; ducks who were sick on the previous day become recovering; and the ducks who were recovering on the previous day become healthy. Prove that the epidemic will eventually stop.

5. Find all positive integers a, b and c such that

$$\operatorname{lcm}(a, b) = \operatorname{lcm}(a + c, b + c).$$

6. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$, such that

$$f(xf(y) - yf(x)) = f(xy) - xy$$

   holds for all real numbers x and y.

7. Let N be a positive integer and $a_1, a_2, \ldots, a_N$ be integers between 2 and 500 inclusive. Suppose for any $k = 1, \ldots, N$ the number $a_1 a_2 .. a_k + 1$ is a perfect square. Prove that $N \leq 10^{50}$.

8. Let n be a positive integer and let $P(x)$ be a polynomial of degree n with real, non-negative coefficients. Suppose that the leading coefficient and the constant coefficient of P are both equal to 1. Prove that if all roots of P are real, then for all $x \geq 0$

$$P(x) \geq (x+1)^n.$$

Battle begins at 14:15.

Each problem is worth 12 points.

The use of electronic devices and literature is strictly prohibited.