International Math Battle Tournament

Bovec, 2026

Day 4

1. Let ABC be a triangle. Let D and E be two points such that B, D, E and C lie on the same line in this order and $2DE = BC$. The lines perpendicular to BC through D and E intersect segments AB and AC at points P and Q, respectively. Let X be the point on the arc PAQ such that $XB = XC$.

   Prove that the line bisecting XB and XC also bisects PQ.

2. Find all integers $a > 1$ for which there exist infinitely many pairs of relatively prime positive integers (m, n) such that $m \mid a^n - 1$ and $n \mid a^m - 1$.

3. Let $d \geq 2$ be a positive integer. Define the infinite sequence $a_1, a_2, \ldots$ by $a_1 = 1$ and

$$a_{n+1} = a_n^d + 1$$

   for all positive integers n. Determine all pairs of positive integers (x, y) such that $a_x \mid a_y$.

4. Find the smallest positive real number r with the following property: For every set $\{v_1, v_2, \ldots, v_{2026}\}$ of 2026 unit vectors in $\mathbb{R}^2$, a point P can be found in the plane, such that for each subset S of $\{1, 2, \ldots, 2026\}$, the sum

$$\sum_{s \in S} v_s$$

   lies in the disc with radius r, centered at P.

5. Determine the largest real number C such that the inequality

$$\frac{x_1 + x_2^{20}}{x_1 + x_3^{26}} + \frac{x_2 + x_3^{20}}{x_2 + x_4^{26}} + \cdots + \frac{x_{2026} + x_1^{20}}{x_{2026} + x_2^{26}} \geq C$$

   holds for all real numbers $x_1, x_2, \ldots, x_{2026} \in (0, 1)$.

6. Let $n \geq 3$ be a positive integer. There are n boxes $A_1, A_2, \ldots, A_n$, each box A_i containing a_i stones with $a_1 + a_2 + \cdots + a_n = 3n$. A move consists of the following operation:

   Choose a box, empty it, and distribute all the resulting stones among the n boxes (including the chosen box) such that for every two boxes the number of stones added to those boxes differ by at most 1.

   For a distribution $a_1, \ldots, a_n$, we define $f(a_1, \ldots, a_n)$ as the least number of moves required to get all the stones into a single box. Let M be the maximum of $f(a_1, \ldots, a_n)$ for all possible distributions $a_1, \ldots, a_n$ of $3n$ stones. Determine M and all distributions $a_1, \ldots, a_n$ for which $f(a_1, \ldots, a_n) = M$.

7. Let ABC be an acute-angled triangle. Let D be the intersection of tangents at points B and C to the circumcircle of triangle ABC. Let M and N be the midpoints of sides AB and AC respectively. Let the circumcircles of triangles DBM and DCN intersect again at the point P, which lies inside the triangle ABC. The segment PD intersects the side BC at the point Q.

   Prove that points M, N, P and Q lie on the same circle.

8. Let $n \geq 2$ be a positive integer. We are given $2n - 1$ (not necessarily distinct) subsets of the set $\{1, 2, \ldots, n\}$, each containing 2 elements. Prove that one can choose n of these such that their union contains no more than $\frac{2}{3}n + 1$ elements.

Battle begins at 14:15.

Each problem is worth 12 points.

The use of electronic devices and literature is strictly prohibited.